Corporate Bond Valuation and Hedging with Stochastic Interest Rates and Endogenous Bankruptcy

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Abstract

- This paper analyzes corporate bond valuation and optimal call and default rules when interest rates and firm value are stochastic.

- **Exogenous models & Endogenous models**

Exogenous- the corporate issuer may be forced to default when firm value or asset cash flow fall too low.

Endogenous- suppose that no such covenants exist.

Abstract

Corporate bonds

(Issuer)

Callable

defaultable

(Buyer)

Putable

Convertible

Interest rate and firm value specifications

- Suppose investors can trade continuously in a complete, frictionless, arbitrage-free financial market.

- There exists an equivalent martingale measure $\tilde{\mathcal{P}}$ under which the expected rate of return on all assets at time t is equal to r_t.

Interest rate and firm value specifications

- The interest rate is a nonnegative one-factor diffusion described by the equation

$$dr_t = \mu(r_t, t)\,dt + \sigma(r_t, t)\,d\widetilde{Z}_t,$$

where $\widetilde{Z}$ is a Brownian motion under $\widetilde{\mathscr{P}}$ and μ and σ are continuous and satisfy Lipschitz and linear growth conditions.

(p11)

Lipschitz and linear growth conditions

- For some constant L, μ and σ satisfy

$$|\mu(x,t)-\mu(y,t)|+|\sigma(x,t)-\sigma(y,t)| \leq L|x-y|,$$

$$|\mu(x,t)|+|\sigma(x,t)| \leq L(1+|x|)$$

for all $x, y, t \in \mathscr{R}^{+}$.

Next, consider a firm with a single bond outstanding. The bond has a fixed continuous coupon c and maturity T. Without loss of generality, suppose the par value of the bond is one, and all other values are in multiples of this par value.

Interest rate and firm value specifications

- The value of the firm is equal to the value of its assets, V, independent of its capital structure.

- Firm value evolves according to the equation

$$\frac{dV_t}{V_t} = (r_t - \gamma_t)\,dt + \phi_t d\widetilde{W}_t,$$

where $\widetilde{W}$ is a Brownian motion under $\widetilde{\mathscr{P}}$ with $d\langle \widetilde{W}, \widetilde{Z}\rangle_t = \rho_t\,dt$ and $\gamma_t \geq 0$, $\phi_t > 0$, and $\rho_t \in (-1, 1)$ are deterministic functions of time.

Option and bond valuation

- Pure callable bond

$$P_C = P_t - f_C$$

- Pure defaultable bond

$$P_D = P_t - f_D$$

- Both callable and defaultable bond

$$P_{CD} = P_t - f_{CD}$$

Option and bond valuation

- The filtration $\{\mathcal{F}_t\}$ generated by the paths of the interest rate and firm value.

- The optimal option value at an arbitrary time *t* in the life of the option is

$$\zeta_t \equiv \sup_{t \leq \tau \leq T} \widetilde{E}\left[\beta_{t,\tau}(P_\tau - \kappa(V_\tau, \tau))^+ | \mathscr{F}_t\right],$$

Option and bond valuation

- The optimal option value at an arbitrary time t in the life of the option is

$$\zeta_t \equiv \sup_{t \le \tau \le T} \widetilde{E}\left[\beta_{t,\tau}(P_\tau - \kappa(V_\tau, \tau))^+ | \mathscr{F}_t\right],$$

where $\widetilde{E}[\cdot]$ denotes the expectation under the measure $\widetilde{\mathscr{P}}$, the strike price

$$\kappa(v, t) = k_t,\ v,\ \text{ or } k_t \wedge v,$$

depending on the bond in question, and the discount factor

$$\beta_{t,\tau} \equiv e^{-\int_t^\tau r_s ds}.$$

(Back)

Option and bond valuation

- Under the Markov interest rate specification, the host bond price

$$P_t = \widetilde{E}\left[c \int_t^T \beta_{t,s} ds + 1 \cdot \beta_{t,T} \mid \mathscr{F}_t \right]$$
$$= p_H(r, t)$$

for some function p_H: $\mathscr{R}^+ \times [0, T] \rightarrow \mathscr{R}$.

$P_H(\cdot, t)$ is strictly decreasing and continuous function

(Back)

Option and bond valuation

given $P_t = p$, and $V_t = v$,

$$\zeta_t = f(p, v, t)$$

for some continuous function $f: \mathscr{R}^+ \times \mathscr{R}^+ \times [0, T] \to \mathscr{R}$, satisfying

$$f(p, v, t) \geq (p - \kappa(v, t))^+.$$

Krylov (1980)

Furthermore, the optimal stopping time is

$$\tau = \inf\{t \geq 0: f(P_t, V_t, t) = (P_t - \kappa(V_t, t))^+\}.$$

Theorem 1.

- The following properties hold for all three embedded options.

1. $p^{(1)} > p^{(2)} \Rightarrow f(p^{(1)}, v, t) > f(p^{(2)}, v, t).$
2. $v^{(1)} < v^{(2)} \Rightarrow f(p, v^{(1)}, t) \geq f(p, v^{(2)}, t).$
3. $p^{(1)} \neq p^{(2)} \Rightarrow \frac{f(p^{(2)},v,t)-f(p^{(1)},v,t)}{p^{(2)}-p^{(1)}} \leq 1.$ (*Call delta inequality*)
4. $v^{(1)} \neq v^{(2)} \Rightarrow \frac{f(p,v^{(2)},t)-f(p,v^{(1)},t)}{v^{(2)}-v^{(1)}} \geq -1.$ (*Put delta inequality*)

No-Crossing Property

Let $\left(r_\tau^{(1)}\right)_{\tau \geq t}$, $\left(r_\tau^{(2)}\right)_{\tau \geq t}$ denote two short rate processes with the same diffusion process but different initial rates, $r_t^{(1)} \leq r_t^{(2)}$. The no-crossing property proved in Karatzas and Shreve (1987) demonstrates that

$$\tilde{P}\left[r_t^{(1)} \leq r_t^{(2)}, 0 \leq t < \infty\right] = 1.$$

Corollary & Lemma

- Let $\beta_t \equiv \underline{\beta_{0,t}} = e^{-\int_0^t r_s ds}$

Corollary 1. *Let $\beta_t^{(1)}$ and $\beta_t^{(2)}$ be the discount factor processes corresponding to initial interest rates $r_0^{(1)}$ and $r_0^{(2)}$, respectively. Then*

$$r_0^{(1)} < r_0^{(2)} \Rightarrow \beta_t^{(1)} > \beta_t^{(2)}, \widetilde{\mathscr{P}} - a.s.\ \forall\ 0 < t < \infty. \tag{23}$$

Proof. From Proposition 2, we have $r_s^{(1)} \leq r_s^{(2)}$, $\forall\ 0 \leq s \leq t$. The paths of $r^{(1)}$ and $r^{(2)}$ are continuous, so there exists a neighborhood around $t = 0$ on which $r^{(1)} < r^{(2)}$. Consequently, $e^{-\int_0^t r_s^{(1)} ds} > e^{-\int_0^t r_s^{(2)} ds}$. ■

Corollary & Lemma

The monotonicity of the host bond price in level of the interest rate implies:

Corollary 2. $r_0^{(1)} \leq r_0^{(2)} \Rightarrow P_t^{(1)} \geq P_t^{(2)}, \widetilde{\mathscr{P}} - a.s.\ \forall\ 0 \leq t \leq T.$

Combining Corollaries 1 and 2 yields:

Corollary 3. $r_0^{(1)} < r_0^{(2)} \Rightarrow \beta_t^{(1)} P_t^{(1)} > \beta_t^{(2)} P_t^{(2)}, \widetilde{\mathscr{P}} - a.s.\ \forall\ 0 \leq t \leq T.$

Corollary & Lemma

Under the firm value specification

$$V_t = V_0 \cdot e^{\int_0^t r_u \, du - \int_0^t \gamma_u \, du - \frac{1}{2} \int_0^t \phi_u^2 \, du + \int_0^t \phi_u \, d\widetilde{W}_u}.$$

It follows that:

Corollary 4. $r_0^{(1)} < r_0^{(2)} \Rightarrow V_t^{(1)} < V_t^{(2)}, \widetilde{\mathscr{P}} - a.s. \; \forall \; 0 < t \leq T.$

Corollary & Lemma

Lemma 1. $r_0^{(1)} \leq r_0^{(2)} \Rightarrow \widetilde{E}[\beta_t^{(2)} P_t^{(2)} - \beta_t^{(1)} P_t^{(1)}] \geq P_0^{(2)} - P_0^{(1)}, \ \forall\ 0 \leq t \leq T.$

Proof. Define the $\widetilde{\mathscr{P}}$-martingale βP^* by

$$\beta_t P_t^* \equiv \widetilde{E}\left[c \int_0^T \beta_s \, ds + 1 \cdot \beta_T | \mathscr{F}_t \right], \ \forall\ 0 \leq t \leq T.$$

Note that

$$\beta_t P_t = \widetilde{E}\left[c \int_t^T \beta_s \, ds + 1 \cdot \beta_T | \mathscr{F}_t \right],$$

so

$$\beta_t P_t^* = \beta_t P_t + c \int_0^t \beta_s \, ds.$$

(p10)

Corollary & Lemma

Rearranging,

$$\beta_t P_t - P_0 = \beta_t P_t^* - c\int_0^t \beta_t\,dt - P_0$$

$$\Rightarrow \widetilde{E}[\beta_t P_t] - P_0 = -\widetilde{E}\left[c\int_0^t \beta_s\,ds\right].$$

Corollary 1 implies that

$$\widetilde{E}\left[c\int_0^t \beta_s^{(1)}\,ds\right] \geq \widetilde{E}\left[c\int_0^t \beta_s^{(2)}\,ds\right],$$

and the result follows.

Proof of Theorem 1.

1. $p^{(1)} > p^{(2)} \Rightarrow f(p^{(1)}, v, t) > f(p^{(2)}, v, t).$

$p^{(1)} > p^{(2)} \Rightarrow r^{(1)} < r^{(2)}$

Let τ be the optimal stopping time given the state at time t is $P_t = p^{(2)}$

$$f(p^{(1)}, v, t) - f(p^{(2)}, v, t) \geq \widetilde{E}\left[\beta_{t,\tau}^{(1)}\left(P_\tau^{(1)} - \kappa\left(V_\tau^{(1)}, \tau\right)\right)^+ - \beta_{t,\tau}^{(2)}\left(P_\tau^{(2)} - \kappa\left(V_\tau^{(2)}, \tau\right)\right)^+ | \mathcal{F}_t\right] > 0.$$

$r^{(1)} < r^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} > \beta_{t,\tau}^{(2)}$, and $P_\tau^{(1)} \geq P_\tau^{(2)}$ $V_\tau^{(2)} \geq V_\tau^{(1)}$

Proof of Theorem 1.

$$2. \quad v^{(1)} < v^{(2)} \Rightarrow f(p, v^{(1)}, t) \geq f(p, v^{(2)}, t).$$

Consider the cases $\kappa(V_t, t) = V_t$ and $\kappa(V_t, t) = k_t \wedge V_t$

By corollary 4, $V_s^{(1)} < V_s^{(2)}, \forall s \in [t, T]. \Rightarrow \kappa(V_\tau^{(1)}, \tau) \leq \kappa(V_\tau^{(2)}, \tau)$

The feasibility of τ as a stopping time for the state $P_t = p$ and $V_t = v^{(1)}$ implies that

$$f(p, v^{(1)}, t) - f(p, v^{(2)}, t) \geq \widetilde{E}\big[\beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(1)}, \tau\big)\big)^+ - \beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(2)}, \tau\big)\big)^+ | \mathscr{F}_t\big] \geq 0.$$

Proof of Theorem 1.

3. $p^{(1)} \neq p^{(2)} \Rightarrow \frac{f(p^{(2)},v,t)-f(p^{(1)},v,t)}{p^{(2)}-p^{(1)}} \leq 1.$ (*Call delta inequality*)

We let $p^{(1)} > p^{(2)}$, $r^{(1)} < r^{(2)}$ and prove that $f(p^{(2)}, v, t) - f(p^{(1)}, v, t) \geq p^{(2)} - p^{(1)}$

Let τ be the optimal stopping time for $p^{(1)}$

$$
\begin{aligned}
& f(p^{(2)}, v, t) - f(p^{(1)}, v, t) \\
& \geq \widetilde{E}[\beta^{(2)}_{t,\tau}(P^{(2)}_{\tau} - \kappa(V^{(2)}_{\tau}, \tau))^{+} - \beta^{(1)}_{t,\tau}(P^{(1)}_{\tau} - \kappa(V^{(1)}_{\tau}, \tau))^{+} | \mathcal{F}_t] \\
& = \widetilde{E}\left\{[\beta^{(2)}_{t,\tau}(P^{(2)}_{\tau} - \kappa(V^{(2)}_{\tau}, \tau))^{+} - \beta^{(1)}_{t,\tau}(P^{(1)}_{\tau} - \kappa(V^{(1)}_{\tau}, \tau))] \cdot 1_{(P^{(1)}_{\tau} > \kappa(V^{(1)}_{\tau}, \tau))} | \mathcal{F}_t\right\} \\
& \geq \widetilde{E}\left\{[\beta^{(2)}_{t,\tau}(P^{(2)}_{\tau} - \kappa(V^{(2)}_{\tau}, \tau)) - \beta^{(1)}_{t,\tau}(P^{(1)}_{\tau} - \kappa(V^{(1)}_{\tau}, \tau))] \cdot 1_{(P^{(1)}_{\tau} > \kappa(V^{(1)}_{\tau}, \tau))} | \mathcal{F}_t\right\} \\
& = \widetilde{E}\left\{[\beta^{(2)}_{t,\tau} P^{(2)}_{\tau} - \beta^{(1)}_{t,\tau} P^{(1)}_{\tau}] \cdot 1_{(P^{(1)}_{\tau} > \kappa(V^{(1)}_{\tau}, \tau))} | \mathcal{F}_t\right\} + \widetilde{E}\left\{[\beta^{(1)}_{t,\tau}\kappa(V^{(1)}_{\tau}, \tau) - \beta^{(2)}_{t,\tau}\kappa(V^{(2)}_{\tau}, \tau)] \cdot 1_{(P^{(1)}_{\tau} > \kappa(V^{(1)}_{\tau}, \tau))} | \mathcal{F}_t\right\} \\
& \geq \widetilde{E}\left\{[\beta^{(2)}_{t,\tau} P^{(2)}_{\tau} - \beta^{(1)}_{t,\tau} P^{(1)}_{\tau}] \cdot 1_{(P^{(1)}_{\tau} > \kappa(V^{(1)}_{\tau}, \tau))} | \mathcal{F}_t\right\} \\
& \geq \widetilde{E}[\beta^{(2)}_{t,\tau} P^{(2)}_{\tau} - \beta^{(1)}_{t,\tau} P^{(1)}_{\tau} | \mathcal{F}_t] \\
& \geq p^{(2)} - p^{(1)}.
\end{aligned}
$$

$\left\{P^{(1)}_{\tau} > \kappa(V^{(1)}_{\tau}, \tau)\right\} \subseteq \left\{P^{(1)}_{\tau} > \kappa(V^{(1)}_{\tau}, \tau)\right\}$

$\beta^{(2)}_{t,\tau} P^{(2)}_{\tau} - \beta^{(2)}_{t,\tau} P^{(2)}_{\tau} < 0$

Proof of Theorem 1.

4. $v^{(1)} \neq v^{(2)} \Rightarrow \frac{f(p,v^{(2)},t)-f(p,v^{(1)},t)}{v^{(2)}-v^{(1)}} \geq -1$. (*Put delta inequality*)

We let $v^{(2)} > v^{(1)}$ and prove that $f(p, v^{(2)}, t) - f(p, v^{(1)}, t) \geq v^{(1)} - v^{(2)}$. Let τ be the optimal stopping time for $v^{(1)}$. Then τ is a feasible stopping time for $v^{(2)}$.

$$
\begin{aligned}
f(p, v^{(2)}, t) - f(p, v^{(1)}, t) &\geq \widetilde{E}\big[\beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(2)}, \tau\big)\big)^+ \\
&\quad - \beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(1)}, \tau\big)\big)^+ | \mathscr{F}_t\big] \\
&= \widetilde{E}\Big\{\big[\beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(2)}, \tau\big)\big)^+ - \beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(1)}, \tau\big)\big)\big] \\
&\qquad \cdot 1_{\left(P_\tau > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} | \mathscr{F}_t\Big\} \\
&\geq \widetilde{E}\Big\{\big[\beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(2)}, \tau\big)\big) - \beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(1)}, \tau\big)\big)\big] \\
&\qquad \cdot 1_{\left(P_\tau > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} | \mathscr{F}_t\Big\} \\
&= \widetilde{E}\Big\{\big[\beta_{t,\tau}\big(\kappa\big(V_\tau^{(1)}, \tau\big) - \kappa\big(V_\tau^{(2)}, \tau\big)\big)\big] \cdot 1_{\left(P_\tau > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} | \mathscr{F}_t\Big\} \\
&\geq \widetilde{E}\big[\beta_{t,\tau}\big(\kappa\big(V_\tau^{(1)}, \tau\big) - \kappa\big(V_\tau^{(2)}, \tau\big)\big) | \mathscr{F}_t\big] \\
&\geq \widetilde{E}\big[\beta_{t,\tau}\big(V_\tau^{(1)} - V_\tau^{(2)}\big) | \mathscr{F}_t\big] \\
&= e^{-\int_t^\tau \gamma_u \, du}\big(v^{(1)} - v^{(2)}\big) \\
&\geq v^{(1)} - v^{(2)}.
\end{aligned}
$$

Proposition 1. *The values of the different embedded options relate as follows.*

$$f_C(p,v,t) \vee f_D(p,v,t) \le f_{CD}(p,v,t) \le f_C(p,v,t) + f_D(p,v,t).$$

Proof of Proposition 1. The first inequality is obvious. We establish the second inequality as follows.

$$\begin{aligned}
f_{CD}(p,v,t) &= \sup_{t\le\tau\le T} \widetilde{E}\big[\beta_{t,\tau}(P_\tau - k_\tau \wedge V_\tau)^+ | \mathscr{F}_t\big] \\
&= \sup_{t\le\tau\le T} \widetilde{E}\big[\beta_{t,\tau}\big((P_\tau - k_\tau)^+ \vee (P_\tau - V_\tau)^+\big) | \mathscr{F}_t\big] \\
&\le \sup_{t\le\tau\le T} \widetilde{E}\big[\beta_{t,\tau}\big((P_\tau - k_\tau)^+ + (P_\tau - V_\tau)^+\big) | \mathscr{F}_t\big] \\
&\le \sup_{t\le\tau\le T} \widetilde{E}\big[\beta_{t,\tau}(P_\tau - k_\tau)^+ | \mathscr{F}_t\big] + \sup_{t\le\tau\le T} \widetilde{E}\big[\beta_t^\tau (P_\tau - V_\tau)^+ | \mathscr{F}_t\big] \\
&= f_C(p,v,t) + f_D(p,v,t).
\end{aligned}$$

Optimal Call and Default Policies

Theorem 2. *Let $t \in [0, T)$ and $v > 0$. If there is any bond price p such that it is optimal to exercise the embedded option at time t given $P_t = p$ and $V_t = v$, then there exists a critical bond price $b(v, t) > \kappa(v, t)$ such that it is optimal to exercise the option if and only if $p \geq b(v, t)$.*

For the proofs of Theorems 2–4, note that the continuation region for each option is the open set

$$U \equiv \left\{(p, v, t) \in \mathscr{R}^+ \times \mathscr{R}^+ \times [0, T]\colon f(p, v, t) > (p - \kappa(v, t))^+\right\}.$$

In addition, note that for all $t \in [0, T)$, $f(p, v, t) > 0$.

Theorem 2.

Proof of Theorem 2. Suppose it is optimal to continue at p_1 and $p_1 > p_2$. We show that it is then optimal to continue at p_2. Using the call delta inequality, we have

$$f(p_2, v, t) \geq f(p_1, v, t) + p_2 - p_1 > (p_1 - \kappa(v, t))^+ + p_2 - p_1 \geq p_2 - \kappa(v, t).$$

In addition, $f(p_2, v, t) > 0$, so

$$f(p_2, v, t) > (p_2 - \kappa(v, t))^+.$$

Let $b(v, t)$ be the supremum of p such that $(p, v, t) \in U$. The point $(b(v, t), v, t)$ cannot lie in U because U is open, so $f(b(v, t), v, t) = b(v, t) - \kappa(v, t) > 0$, which implies $b(v, t) > \kappa(v, t)$. ■

Theorem 3. *Let $t \in [0, T)$ and $p > 0$.*

1. *For the pure defaultable bond, there exists a critical firm value $v_D(p, t) < p$ such that, at time t, given $P_t = p$ and $V_t = v$, it is optimal to default if and only if $v \le v_D(p, t)$.*
2. *For the callable defaultable bond, there exists a critical firm value $v_{CD}(p, t)$, satisfying $v_{CD}(p, t) \le k_t$ and $v_{CD}(p, t) < p$, such that, at time t, given $P_t = p$ and $V_t = v$, it is optimal to default if and only if $v \le v_{CD}(p, t)$. In addition, if there exists any firm value v at which it is optimal to call, then there exists a critical firm value $\bar{v}_{CD}(p, t) \ge k_t$ such that it is optimal to call if and only if $v \ge \bar{v}_{CD}(p, t)$.*

Proof of Theorem 3. 1. Note that it must be optimal to default at $v = 0$. Suppose it is optimal to continue at v_1 and $v_1 < v_2$. We show that it is then optimal to continue at v_2. Using the put delta inequality,

$$f(p, v_2, t) \geq f(p, v_1, t) + v_1 - v_2 > (p - v_1)^+ + v_1 - v_2 \geq p - v_2, \tag{58}$$

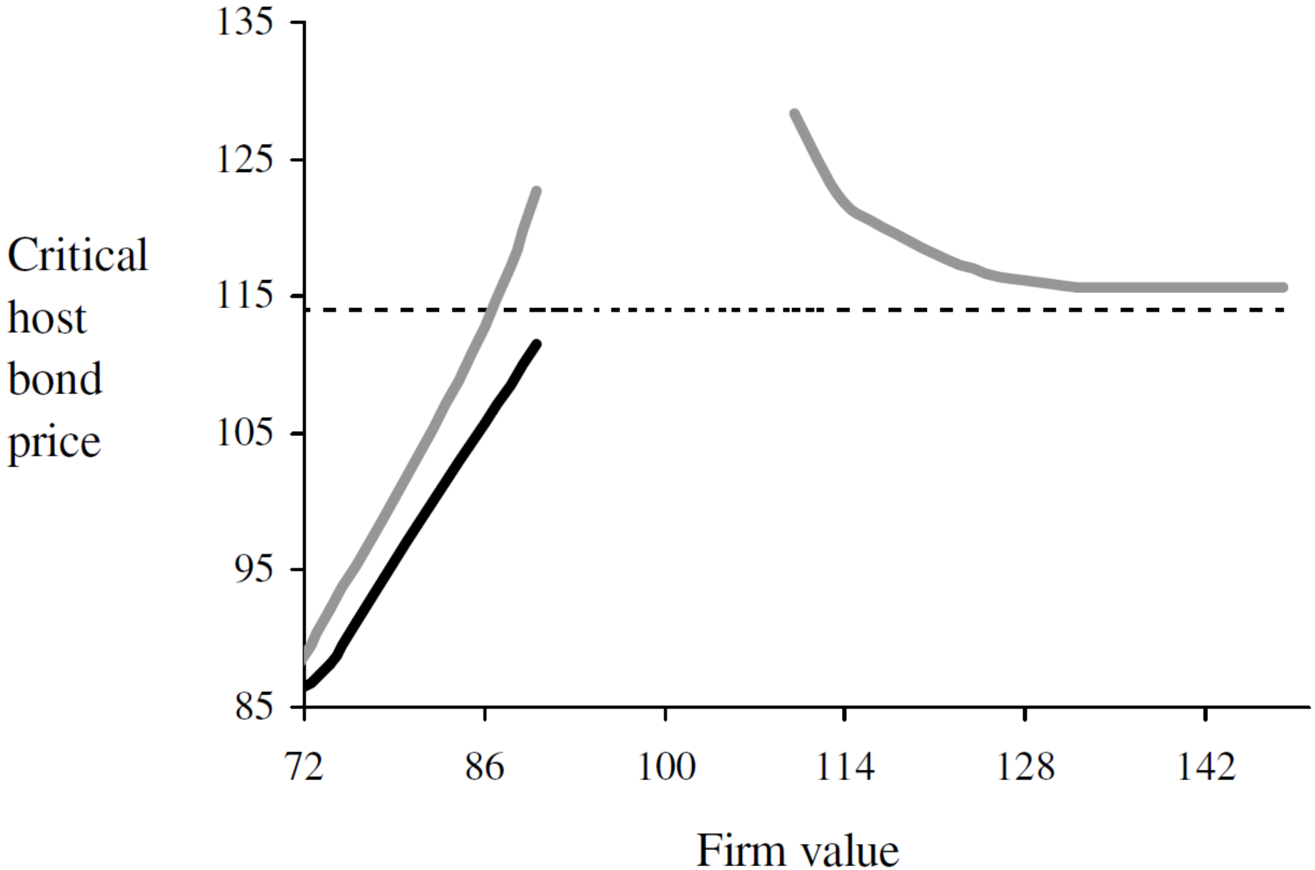
and thus $f(p, v_2, t) > (p - v_2)^+$. Let $v_D(p, t)$ be the infimum of v such that $(p, v, t) \in U$. Since $f(p, v_D(p, t), t) > 0$, $v_D(p, t) < p$.

2. First, suppose it is optimal not to default at v_1 and $v_1 < v_2$. We show that it is then also optimal not to default at v_2. From the put delta inequality,

$$
\begin{aligned}
f(p, v_2, t) &\geq f(p, v_1, t) + v_1 - v_2 > (p - v_1 \wedge k_t)^+ + v_1 - v_2 \\
&\geq p - v_2,
\end{aligned}
$$

and thus $f(p, v_2, t) > (p - v_2)^+$.
Note that it must be optimal to default at $v = 0$. Therefore, there exists a critical value $v_{CD}(p, t)$ such that it is optimal to default $\forall v, v \leq v_{CD}(p, t)$.

Next, suppose it is optimal to call at v_1, and $v_1 < v_2$. We show that then it is then optimal to call at v_2. Note that $k_t \leq v_1$ must hold. Now, on one hand, $f(p, v_2, t) \geq p - k_t \wedge v_2 = p - k_t$. On the other hand, from part 2 of Theorem 1, $f(p, v_2, t) \leq f(p, v_1, t) = p - k_t$. Let $\bar{v}_{CD}(p, t) \geq k_t$ be the minumum of v such that it is optimal to call at (p, v, t). ■



Critical host bond price
135
125
115
105
95
85
72
86
100
114
128
142
Firm value

Theorem 4. *For each* $t \in [0, T)$,

1. $v_1 < v_2 \Rightarrow b_D(v_1, t) \le b_D(v_2, t)$.
2. $p_1 < p_2 \Rightarrow v_D(p_1, t) \le v_D(p_2, t)$.
3. $v_1 < v_2 \le k_t \Rightarrow b_{CD}(v_1, t) \le b_{CD}(v_2, t)$.
4. $k_t < v_1 < v_2 \Rightarrow b_{CD}(v_1, t) \ge b_{CD}(v_2, t)$.
5. $v \le k_t \Rightarrow b_{CD}(v, t) \ge b_D(v, t)$.
6. $v > k_t \Rightarrow b_{CD}(v, t) \ge b_C(v, t)$.

Proof of Theorem 4. 1. Suppose $0 < p < b_D(v_1, t)$. Then $p < b_D(v_2, t)$ as well:

$$f(p, v_2, t) \geq f(p, v_1, t) + v_1 - v_2 > p - v_1 + v_1 - v_2 = p - v_2.$$

2. Suppose $v > v_D(p_2, t)$. Then $v > v_D(p_1, t)$ as well:

$$f(p_1, v, t) \geq f(p_2, v, t) + p_1 - p_2 > p_2 - v + p_1 - p_2 \geq p_1 - v.$$

3. The proof is essentially the same as that in part 1.
4. Suppose $0 < p < b_{CD}(v_2, t)$. Then $p < b_{CD}(v_1, t)$ as well:

$$f(p, v_1, t) \geq f(p, v_2, t) > g(p, v_2, t) = (p - k_t)^+ = g(p, v_1, t).$$

5. If $p < b_D(v, t)$, then $f_{CD}(p, v, t) \geq f_D(p, v, t) > p - v = p - v \wedge k_t$, so $p < b_{CD}(v, t)$.
6. If $p < b_C(v, t)$, then $f_{CD}(p, v, t) \geq f_C(p, t) > p - k_t = p - v \wedge k_t$, so $p < b_{CD}(v, t)$. ■

(Th.1)

Pure Convertible Bond

- To keep problems simple, we follow Acharya and Carpenter(2002) by assuming the market value of the firm

$$V_t = N_C C_t + N_0 S_t^{PC}$$

- The outstanding shares of the stock increase by

$$\Delta N (\equiv \eta N_C)$$

Pure Convertible Bond

- Ignoring the effects of tax benefits and bankruptcy cost, after conversion:

$$V_t = (N_0 + \Delta N) S_t^{AC},$$

- The after-conversion stock price

$$S_t^{AC} = \frac{V_t}{N_0 + \Delta N}.$$

Pure Convertible Bond

- The value to convert a bond into η shares of stocks is

$$\eta S_t^{AC} = \frac{\eta}{N_0 + \Delta N} V_t \equiv z V_t,$$

- Pure convertible bond

$$P_{PC} = P_t + f_{PC}$$

- The optimal stopping time

$$\tau = \inf\{t \geq 0: f(P_t, V_t, t) = (zV_t - P_t)^+\}$$

DCC Bond

Corporate bonds

(Issuer)

Callable

defaultable

(Buyer)

Putable

Convertible

Numerical Methods

- Firm value – BTT tree
- Interest rate – Hull white tree model
- Backward induction

Numerical Methods

- Value= max(min(cont ,min(Vt,K)-Pt) ,z*Vt-Pt);